*Journal of Statistical Physics, Vol. 114, Nos. 1/2, January 2004 (© 2004)*

# A Gibbs-Like Measure for Single-Time, Multi-Scale Energy Transfer in Stochastic Signals and Shell Model of Turbulence

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*Received November 26, 2002; accepted June 16, 2003*

A Gibbs-like approach for simultaneous multi-scale correlation functions in random, time-dependent, multiplicative processes for the turbulent energy cascade is investigated. We study the *optimal* log-normal Gibbs-like distribution able to describe the subtle effects induced by non-trivial time dependency on both single-scale (structure functions) and multi-scale correlation functions. We provide analytical expression for the general multi-scale correlation functions in terms of the two-point correlations between multipliers and we show that the log-normal distribution is already accurate enough to reproduce quantitatively many of the observed behavior. The main result is that non-trivial time effects renormalize the Gibbs-like *effective* potential necessary to describe single-time statistics. We also present a generalization of this approach to more general, non log-normal, potentials. In the latter case one obtains a formal expansion of both structure functions and multi-scale correlations in terms of cumulants of all orders.

**KEY WORDS:** Turbulence; multifractals; stochastic processes; shell models.

## 1. INTRODUCTION

Small scales, three-dimensional, turbulent fluctuations are sustained by the energy cascade mechanism: energy is injected at large scales, $L_0$, and dissipated at small scales, $\eta$. The statistical properties of the energy transfer throughout the range of scales going from $L_0$ to $\eta$ (inertial range) are

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thought to possess highly non-trivial features, mainly, but not only, summarized by the presence of anomalous scaling, i.e., $p$th order velocity structure functions, $S_p(r) = \langle (v(r) - v(0))^p \rangle$ have a power law scaling with anomalous exponents, $S_p(r) \sim (r/L_0)^{\zeta(p)}$. The existence of a net, direct, energy flux in the inertial range is a clear indication of the out-of-equilibrium nature of turbulent flows. The exact symmetry of the inertial Navier–Stokes terms with respect to direct or inverse energy transfer is explicitly broken by the existence of an energy source at large scales and an energy sink at small scales. The energy cascade process has been often, and fruitfully, described in terms of a multi-step fragmentation process describing the tendencies of inertial range eddies to break in smaller and smaller eddies, following the celebrated Richardson scenario.[1] The spatio-temporal complexity of the fragmentation process has been successfully described by ref. 5 using the multifractal language, which have proved able to reproduce qualitatively and quantitatively single-scale, multi-scale and multi-time multi-scale velocity correlation functions.[3–6]

By using the multifractal language one can assume that the velocity difference on scale $r_2$ is linked to the velocity difference at scale $r_1 \geqslant r_2$ by the equation

$$\delta v(r_2) = M(r_2, r_1)\, \delta v(r_1), \tag{1}$$

where $M(r_2, r_1)$ is a suitable random variable. Moreover, one often limits the complexity by assuming almost uncorrelated multipliers:

$$M(r_1, r_3) = M(r_1, r_2)\, M(r_2, r_3)$$

for any choices of $r_1 < r_2 < r_3$ in the inertial range. The above assumptions lead to the concept of random multiplicative process as a possible way to characterize the turbulent, multifractal, velocity field. In ref. 2 a Gibbs measure for the random multiplicative process has been introduced in order to compute, self consistently, the anomalous scaling exponents for a class of Shell models. Recently, the idea discussed in ref. 2 have been further developed in ref. 7 by studying some new proposals on how to use a Gibbs-like approach to describe energy fluctuations in the inertial range. According to the Kolmogorov hypothesis of local interactions in Fourier space, one expects short range correlations between multipliers, while velocity fields may still show long-range correlations. On one hand, the simplest phenomenological multifractal description able to capture the correct anomalous scaling for single-scale structure functions is based on the assumption of a complete uncorrelated multiplicative process.[8] On the other hand, the assumption of completely independence between multipliers is untenable because in disagreement with some theoretical and experimental results. In this paper, we try to perform a step toward a more realistic description of turbulent energy cascade by studying also deterministic and stochastic processes with correlation between multipliers. Correlations are introduced by a non-trivial time evolution of multipliers in a *self-consistent* way, i.e., by imposing that time and spatial fluctuations are linked as dictated by the non-linear terms of the Navier–Stokes equations. The main goal is to obtain the optimal *effective Gibbs* potential able to describe the simultaneous fluctuations of the velocity field at all scales.

We study the problem in both deterministic models of turbulent energy cascade (Shell Models)[9, 10] and in stochastic, time-dependent, multiplicative processes. Let us recall that shell models are the simplest deterministic models whit anomalous multi-time and multi-scale velocity correlation functions. Moreover, shell models possess the special feature to have no sweeping terms, i.e., also temporal properties of the energy cascade can be tested.

The paper is organized as follows. First we briefly recall the main features of Shell models and we define the set of observable we want to describe within the Gibbs approach. Then, we discuss the simplest, log-normal, approximation for the Gibbs-potential. Log-normal potential enjoys the properties to be "exactly solvable," any structure functions and multi-scale correlation function possess an explicit expression in terms of the two-point *spin-spin* correlation function (see below). We present some evidences that already the simple log-normal approximation is able to capture many of the observed behaviors for both stochastic process and shell models. Further, we discuss how to generalize the approach to a more general, i.e., non log-normal, potential, and finally, we conclude with comments on possible future works.

## 2. THE GIBBS ENSEMBLE

Shell models describe the energy turbulent transfer on a set of scales (shells) in the Fourier space, $k_n = k_0 \lambda^n$, where $k_0$ is the smallest wavenumber and $\lambda$ is the inter-shell ratio, usually set to 2. Velocity shell variables, $u_n(t)$ are complex numbers representing velocity fluctuations, $\delta_{r_n} v$, over a scale $r_n = k_n^{-1}$, with $n = 0,\dots, N$. Among all possible shell models a very popular one is the Sabra model[11] an improved version of the GOY model.[9] The model is:

$$(d/dt + \nu k_n^2)\, u_n = i k_n (u_{n+2} u_{n+1}^* + b u_{n+1} u_{n-1}^* - c u_{n-1} u_{n-2}) + f_n \qquad (2)$$

where $\nu$ is the kinematical viscosity, $b, c$ are free parameters fixed by requiring that energy and helicity are inviscid invariants of the models and

$f_n$ is the external forcing with supports only on large scales (small shell indexes). The existence of only local interactions between shells allows to have highly non-trivial time properties, i.e., the model is a reliable approximation of velocity evolution in a quasi-Lagrangian reference frame. The model is known to possess realistic multi-time and multi-scale correlation functions, including anomalous inertial range scaling and dissipative anomaly, for a review see refs. 5, 8, and 10. Typical observable checking single-scale Probability Density Functions (PDF) are given by structure functions:

$$S_p(k_n) = \langle |u_n|^p \rangle. \tag{3}$$

On the other hand, multi-scale single-time observable which will play a relevant role in the following are the two-scale correlation function:

$$F_{p,q}(k_n, k_{n+m}) = \langle |u_n|^p \, |u_{n+m}|^q \rangle. \tag{4}$$

Usually, multifractal phenomenology is based on uncorrelated multipliers, $M(k_n, k_{n+m})$, connecting the two shell velocity at scales $k_n, k_{n+m}$ as $u_{n+m} = M(k_n, k_{n+m})\, u_n$. This simple scenario leads naturally to pure, anomalous, power law scaling for the structure functions (3),

$$S_p(k_n) \sim k_n^{-\zeta(p)}$$

and to pure *fusion-rules* predictions for the two scales correlation:[4–6]

$$F_{p,q}(k_n, k_{n+m}) \propto (k_{n+m}/k_n)^{-\zeta(q)}\, k_n^{-\zeta(p+q)}. \tag{5}$$

The numerical integration of shell model equations (2) shows that the behavior of the multi-scale correlation function, $F_{p,q}(k_n, k_{n+m})$, predicted by (5) is true only asymptotically for *very large* scale separation $k_{n+m}/k_n \to \infty$, while important deviations are detected for scale separation, $k_{n+m}/k_n \sim O(1)$. The origin of such deviation can easily be understood if one looks at the typical temporal evolution of the energy contents at different scale. In shell models, energy is transferred down-scale by a burst-like activity, strong coherent energy bumps travel from large-scales to small-scales. Each scale has its typical, fluctuating, eddy-turn-over time, $\tau_n \sim 1/(u_n k_n)$. Energy is transferred from scale $k_n$ to scale $k_N$ in a typical time $\tau_{n,N} = \tau_n - \tau_N$ (with $N > n$). The non-instantaneous, intermittent, propagation of energy from shell to shell is dynamically realized by non-trivial fluctuations of shell model phase variables, $\phi_n$ of (2)—where $u_n = |u_n| \exp^{i\phi_n}$.

The time-delay in the information propagation has important feedback also on single time observable as structure functions and multi-scale correlation functions (see below). This is the physical reason why the understanding of single time statistics calls for the understanding also of multi-time statistics. The scope of this article is to investigate to which extent one may try to build up an *effective* Gibbs-like description capable to incorporate the effects of non-trivial time fluctuations on single-time statistics. In the following we restrict ourself to discuss the probability distribution function of shell amplitude, $|u_n|$. In ref. 7, following the original proposal made in ref. 2, a Gibbs hypothesis has been developed for the simultaneous probability distribution function of the $u_n$ set of shell variable:

$$P(u_1, u_2,..., u_N) \propto e^{-\Phi(u_1, u_2,..., u_N)} \tag{6}$$

where for sufficiently high Reynolds numbers we will suppose the potential, $\Phi$, to become translational invariant in the shell index (independent on the UV and IR boundary condition). As previously said, we also make, following,[2, 3] the further assumption that the potential depends only on ratios between shell variables, i.e., it is an homogeneous function of zero degree. The physical rationale for this assumption stem from the original Kolmogorov remark that energy-cascade is maintained by a local transfer in Fourier space. Introducing the *spin* variables, $\sigma_j = \log_2(\frac{|u_{j-1}|}{|u_j|})$ we may rewrite the Gibbs-hypothesis for the shell amplitudes as:

$$\Phi(|u_1|, |u_2|,..., |u_N|) = \phi(\sigma_1, \sigma_2,..., \sigma_N).$$

Let us remark that the Gibbs-potential has nothing to do with the "equilibrium" distribution obtained by the equipartition of the inviscid invariants even in the limit of $\nu \to 0$. In ref. 7 a very detailed numerical and theoretical analysis on the consequences of such a description has been made, giving strong evidences that the formalism is consistent with the statistical properties of the model. Here, we want to push further this hypothesis by explicitly looking at the possible Gibbs-potential able to reproduce quantitatively and qualitatively the measured structure functions (3) and multi-scale correlation functions (4). A pure uncorrelated multiplicative process on the shell amplitude is described in the Gibbs formalism by a simply infinite temperature Gibbs potential with no interaction between spins:

$$\phi(\sigma_1, \sigma_2,..., \sigma_N) = \sum_{j=1}^{N} V(\sigma_j). \tag{7}$$

Indeed, it is simple to show that in this case we have for the structure functions the exact anomalous scaling:

$$S_p(k_n) = \langle |u_n|^p \rangle \propto \int \prod_j d\sigma_j \exp\left\{ -\left( \sum_{j=1}^{N} V(\sigma_j) + p \log(2) \sum_{j=1}^{n} \sigma_j \right) \right\} \propto k_n^{-\zeta(p)}$$

with $\zeta(p) = -\log_2 \langle 2^{-p\sigma} \rangle$, where with $\langle \cdot \rangle$ we mean the average with respect to the Gibbs measure $\prod d\sigma_j \exp\{-\phi(\sigma_1, \sigma_2,..., \sigma_N)\}$. Similarly, for the multi-scale correlation function we have:

$$F_{p,q}(k_n, k_{n+m}) = \langle |u_n|^p \, |u_{n+m}|^q \rangle = \langle 2^{-(p+q)\sum_{j=1}^{n} \sigma_j - q \sum_{j=n+1}^{n+m} \sigma_j} \rangle, \qquad (8)$$

which correspond to the fusion-rule prediction (5) for *all* scale separation.

On the other hand, as previously said, the pure uncorrelated hypothesis cannot be correct because it does not predict the observed behavior of multi-scale separations for small scale separation, $k_n/k_{n+m} \sim O(1)$. Indeed, two-scales correlation as (4) tends to the fusion-rule prediction (5) only asymptotically—see for example Fig. 2. Thus, it seems necessary to add to the uncorrelated potential (7) some spin-interaction. The simplest analytical way to do it is to stay within all possible interaction in a Gaussian field, i.e., to consider a "correlated" log-normal distribution for the shell variables. As we shall see in the last section, this assumption is not restrictive, most of the qualitative and quantitative results here presented can be extended to more complex probability distribution.

A log-normal uncorrelated process is described by the Gibbs potential: $\phi(\sigma_1, \sigma_2,..., \sigma_N) = \sum_{j=1}^{N} \frac{(\sigma_j - h_0)^2}{(2c^2)}$ where the only two free parameters are the log-mean $h_0$ and the log-variance $c^2$. Correlations among multipliers can be introduced by writing the Gibbs potential as:

$$\phi(\sigma_1, \sigma_2,..., \sigma_N) = \tfrac{1}{2} \sum_{j,i} \sigma_j A_{ji} \sigma_i - h_0/c^2 \sum_j \sigma_j \qquad (9)$$

where now the matrix of interaction is given by $A_{ji} = \delta_{ji}/c^2 - J_{ji}$. Clearly, by taking $J_{ji}$ depending only on the separation between shell indexes, $j-i$, we may describe the most general translational invariant, log-normal, two-body potential. Translational invariance in the spins variable is the counterpart of scaling invariance for the velocity variables.

As previously said, in shell models one cannot simply disentangle the amplitude fluctuations, $|u_n|$, from the phase fluctuations, $\phi_n$, in other words one cannot expect to reproduce *quantitatively* the multi-scale fluctuations without a further explicit introduction, in the Gibbs formalism, also of phase-variables. In order to make the discussion simpler, we will test quantitatively the Gibbs-formalism on a stochastic, time-dependent, multiplicative signal involving only amplitude fluctuations, meant to mimic the amplitude evolution of shell models dynamics. We will go back to comparison with the outcomes of the original deterministic dynamics (2) only to show the ability of the Gibbs formalism to catch the main qualitative behaviors.

The stochastic, time-dependent, multiplicative process is built as follows (see also ref. 12).

We introduce $N$ i.i.d. random variables, $W_j = M(k_{j+1}, k_j)$, one for each shell, describing the uncorrelated instantaneous multipliers connecting amplitudes of shell variables between shells $n$ and $n+1$, i.e., $|u_{n+1}| = W_n\, |u_n|$. The probability of $W$ coincides with the log-normal uncorrelated Gibbs-potential: $P(W_n) \propto \exp\{-\frac{(\sigma_n - h_0)^2}{2c^2}\}$. To generate the time dynamics we proceed as follows. We extract $W_n$ with probability $P(W_n)$ and keep it constant for a time interval $[t, t+\tau_n]$, with $\tau_n = 1/(|u_n|\, k_n)$ being the local instantaneous eddy-turn-over time, Thus, for each scale $k_n$, we introduce a time dependent random process $W_n(t)$ which is piece-wise constant for a random time intervals $[t_n^{(k)}, t_n^{(k)} + \tau_n]$, if $t_n^{(k)}$ is the time of the $k$th jump at scale $n$. The corresponding velocity field at scale $n$, in the time interval $t_n^{(k)} < t < t_n^{(k)} + \tau_n$, is given by the simple multiplicative rule:

$$|u_n(t)| = W_n(t)\, |u_{n-1}(t_n^{(k)})|. \tag{10}$$

What is important to notice is that at each jumping time, $t_n^{(1)}, t_n^{(2)},..., t_n^{(k)},...$, for any scale, $n$, only the local velocity field is updated, i.e., information across different scales propagates with a finite speed. It is easy to realize that the propagation speed is proportional to the characteristic speed of the energy cascade in turbulent flows, i.e., a fluctuations in the multiplier at scale $k_n$ takes a time $T \sim \tau_n - \tau_m$ to propagate down to scale $k_m$. In this way we reproduce the phenomenology of the non-linear evolution of the shell model dynamics: $d/dt\, u_n \propto k_n u_n^2$, which is itself meant to mimic the non-linear evolution of a Navier–Stokes field in quasi-Lagrangian reference frame. This is the simplest stochastic evolution with non-trivial spatial and temporal fluctuations in qualitative agreement with the shell models phenomenology.[12] In the following, whenever we refer to the time evolution of the stochastic process at scale $k_n$ we use the notation $u_n^{(s)}(t)$ to distinguish it from the time evolution of the deterministic shell-model velocity, $u_n(t)$. The link between spatial fluctuation and temporal fluctuations, $\tau_n = 1/(k_n u_n)$, has important feedback even on the single time structure functions, $S_p(k_n)$, and single time multi-scale correlation functions, $F_{p,q}(k_n, k_{n+m})$. In Fig. 1 we plot the scaling exponents of the structure

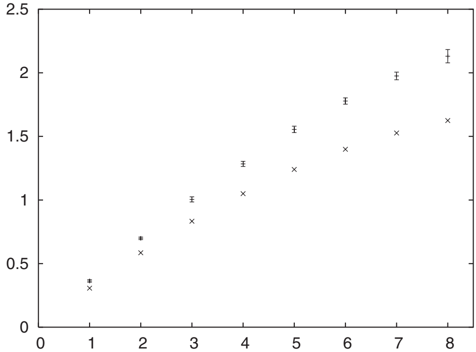


Fig. 1. Comparison between the scaling exponents, $\zeta(p)$, calculated on the simple multiplicative process without time dependencies, ($\times$) and with fluctuating time, (+). Notice the *renormalization* observed in the values of the exponents once fluctuating eddy-turn-over times are switched on.

functions measured on the stochastic field, $S_p^{(s)}(k_n) = \langle |u_n^{(s)}|^p \rangle$ with and without fluctuating eddy-turn-over time. As one can see there is an important "renormalization" effect when eddy-turn-over times fluctuate. The scaling exponents move from the simple uncorrelated value when the time dynamics is trivial to a *renormalized* value when local multipliers are updated with stochastic times. This *renormalization* effect can be understood in terms of the correlation between the fluctuating eddy-turn-over times and the multipliers.[12] This is the first evidence of a relevant effect on the energy cascade introduced by the time dynamics even on single-time observable. In Fig. 2 we also compare the behavior of a single time multi-scale correlation

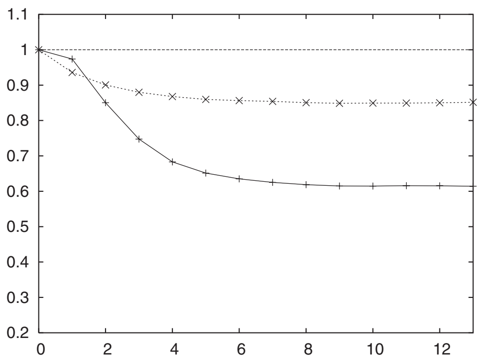


Fig. 2. Normalized multi-scale correlation function $C_{2,2}^{(m)} = \frac{\langle u_n^p u_{n+m}^q \rangle \langle u_n^q \rangle}{\langle u_{n+m}^q \rangle \langle u_n^{p+q} \rangle}$ with $n = 12$ as a function of the scale separation $m$, for shell model (+) and stochastic multiplicative signal with time dependencies ($\times$). The straight line of value 1, corresponds to the trivial case of a multiplicative uncorrelated process without any time-dependency.

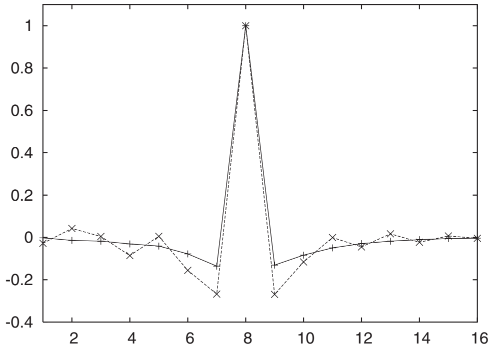


Fig. 3. Two point normalized connected correlation function among multipliers, *spins*, $\langle\sigma_i\sigma_j\rangle_c/\langle\sigma_i\sigma_i\rangle_c$ with $i=8$ at changing the scale separation $j=i-8,\ldots,i+8$ calculated on the shell model ($\times$) and on the time-dependent stochastic signal ($+$). Both $i$ and $j$ are in the inertial range for the shell model simulation.

function, $F_{2,2}(k_n, k_{n+m})$, measured on the stochastic process *with and without* time dependency and in the original shell model (2). It is important to notice that both shell model and the stochastic, time-dependent, process show the same similar slow approach to the asymptotic plateau for the normalized multi-scale correlation function. The above result suggests that the departure measured for small scale separation from the asymptotic fusion-rule prediction (5) is mainly due to non-trivial correlation between multipliers introduced by the time-dynamics.[5]

In Fig. 3 we show the spin-spin correlation, i.e., the correlation among multipliers, for both the time dependent random multiplicative process and the equivalent quantities in the shell model. It is rather clear that time dynamics introduces a correlation among multipliers in a non trivial way. Moreover, it is important to remark that the spin-spin correlation is qualitative similar for both the time dependent random multiplicative process and the shell model. In particular let us remark that the near neighborhood correlation is negative in both cases.

## 3. SOME RIGOROUS RESULTS

In both shell models and the stochastic process all single time correlation functions are determined by the whole spatio-temporal dynamics. We want now investigate the possibility to reproduce the effects of the non-trivial time properties of the energy cascade on the single-time statistics. We try to do it by an *effective*, time independent, Gibbs potential. Therefore, the Gibbs-potential must be seen as a *renormalized* set of time-independent interaction describing the whole set of possible multi-scale *single-time* correlation functions.

In order to keep our discussion as simple as possible we confine here to investigate log-normal distributions. In the next section, we generalize our results for any probability distribution of random multipliers. As discussed in Section 2, we define the random multipliers as the ratio between stochastic fields at neighboring scales: $u_i^{(s)}/u_{i-1}^{(s)} = W_i = 2^{-\sigma_i}$. It follows that the joint probability distribution for a given set of spins/multipliers $\{\sigma_i\}$ is given by:

$$P[\sigma_i] \propto \exp\left\{h_0/c^2 \sum_{i=1}^{N} \sigma_i - \tfrac{1}{2}\sum_{i,j} A_{i,j}\sigma_i\sigma_j\right\}, \tag{11}$$

where $A_{i,j}$ is the spin-spin interaction. We introduce the shorthand notation $\vec{\sigma}$ to denote the $N$-component vector formed by all spins $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ and $\hat{A}$ to denote the interaction matrix. Then, we may rewrite the partition function $\mathscr{Z}(\vec{h}_0)$:

$$\mathscr{Z}(\vec{h}_0) = \int d\vec{\sigma} \exp\left\{-\tfrac{1}{2}\vec{\sigma}\hat{A}\vec{\sigma} + \vec{h}_0 \cdot \vec{\sigma}\right\} \sim \exp^{\frac{1}{2}(\vec{h}_0\hat{A}^{-1}\vec{h}_0)} \tag{12}$$

where the vector $\vec{h}_0$ is made of constant entries for all scales: $\vec{h}_0 \equiv h_0/c^2(1, 1, 1, 1, \ldots)$. Having restricted ourselves to the most general Gaussian distribution it is not surprising that one can work out an explicit formula for the most general multi-scale correlation functions in terms of the two-point connected correlation function, $\langle\sigma_i\sigma_j\rangle_c = \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$, only. In the appendix we present the long straightforward calculation leading to the final expression:

$$C_{p,q}^{(m)} = \frac{\langle u_n^p u_{n+m}^q\rangle\langle u_n^q\rangle}{\langle u_{n+m}^q\rangle\langle u_n^{p+q}\rangle} = e^{\mathscr{T}_{p,q}^m} \tag{13}$$

with

$$\mathscr{T}_{p,q}^m = (\log 2)^2\, pq\left(\sum_{i=1}^{n} \langle\sigma_i\sigma_{n+1}\rangle_c + \langle\sigma_i\sigma_{n+2}\rangle_c + \cdots + \langle\sigma_i\sigma_{n+m}\rangle_c\right). \tag{14}$$

The above formula has a particularly appealing interpretation: deviations of the multi-scale correlation functions from its "multiplicative uncorrelated" fusion-rules prediction (5) is governed by the short range correlation between multipliers. Indeed, in the RHS of (14) the main contribution is carried by the first connected correlation function, $\langle \sigma_n \sigma_{n+1} \rangle_c$, while all the other terms becomes less and less important because they connect spins at larger and larger distances. It turns out that spins are anti-correlated at short distances (see Fig. 3). This is the reason why all normalized multi-scale correlation functions (13) converge to a plateau smaller than unity, i.e., $C^{(m)}_{p,q} < 1$. Moreover, in this log-normal approximation, the coefficients $C^{(m)}_{p,q}$ are symmetrical in $p, q$ something which may be exploited in order to reduce the number of degrees of freedom in closures.[13]

### 3.1. Numerical Tests

Let us now try to check numerically whether the log-normal approximation is in good agreement with the numerics observed in the stochastic signal, $u^{(s)}_n(t)$. As already remarked, the most general log-normal distribution is completely fixed once one defines the $A_{ij}$ coupling matrix and the magnetic field $h_0$ defining the "bare" probability distribution function of multipliers, $h_0 = -\langle \log_2 W \rangle$. Moreover, the coupling matrix is in one-to-one correspondence with the connected two-spins correlation functions:

$$A^{-1}_{ij} = \langle \sigma_i \sigma_j \rangle_c. \tag{15}$$

We have therefore taken as our *best guess* for the coupling potential the expression for $J_{ij}$ obtained from (15) by using the measured two-point correlations $\langle \sigma_i \sigma_j \rangle_c$ in the stochastic evolutions—a plot of $\langle \sigma_i \sigma_j \rangle_c$ is shown in Fig. 3. Then, from this numerical input we can check whether the quantities analytically computable from (11) are in agreement with those measured on the stochastic signal. In Fig. 4 we compare the two multi-scale correlation functions calculated either numerically or from the (11) expression. As on can see the agreement is qualitative and quantitatively very satisfactory. The above results tell us that the most general log-normal distribution, chosen to exactly reproduce the measured two-point correlation function, $\langle \sigma_i \sigma_j \rangle_c$ is also able to reproduce in a quantitative way the multi-scale correlation function with high accuracy, i.e., the log-normal approximation with the potential given in Fig. 3 is a very close approximation of the "effective" single-time probability distribution of the complete time-dependent stochastic process. Also scaling exponents measured on the correlated log-normal potential are in good agreement with those measured numerically on the time-dependent stochastic process (with deviations of order 5% on the 10th order exponent). Furthermore, in the next section, we will show that the log-normal result can be seen as the first term in a systematic expansion in cumulants of the most general probability distributions.

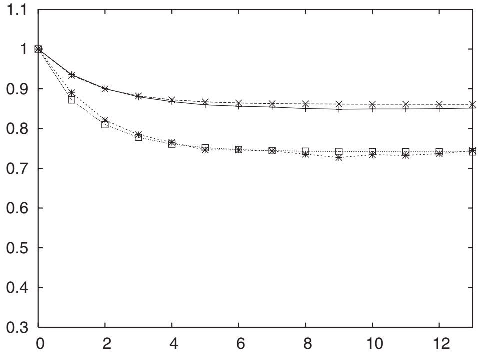


Fig. 4. Comparison between two-scale normalized correlation functions, calculated from the log-normal Gibbs formalism and on the stochastic time-dependent multiplicative process. Top curves: $C^{(m)}_{22}$ from the Gibbs formalism, (+), and for the stochastic process, (×). Bottom curves: $C^{(m)}_{24}$ for the Gibbs formalism, (squares), and for the stochastic process (∗).

## 4. A GENERALIZATION TO NON-GAUSSIAN DISTRIBUTIONS

We want here to present a simple argument showing that the expression (14) obtained within the log-normal approximation can always be seen as the first term of a formal cumulant expansion for the most general potential.

Let us consider the generic interacting potential $\Phi(\sigma_1,..., \sigma_N)$ among the $N$ spins. Where we now may include also three-body and multi-body interactions in $\Phi$ and/or the spins variable can take values also on a discrete set (Ising-like systems). If we go back to the observable we want to control in our multiplicative process, we realize that one may write both structure functions and multi-scale correlation functions as suitable partition function calculated with suitable *external magnetic field*:

$$\langle u_n^p \rangle \propto \sum_{\{\sigma_i\}} \exp\{\vec{\sigma} \cdot \vec{H}_n^p - \Phi(\sigma_1,..., \sigma_N)\} = Z(\vec{H}_n^p) \tag{16}$$

and

$$\langle u_n^p u_{n+m}^q \rangle \propto \sum_{\{\sigma_i\}} \exp\{\vec{\sigma} \cdot \vec{H}_{n,n+m}^{p,q} - \Phi(\sigma_1,..., \sigma_N)\} = Z(\vec{H}_{n,n+m}^{p,q}) \tag{17}$$

where $\vec{H}_n^p$ and $\vec{H}_{n,n+m}^{p,q}$ are site-dependent magnetic fields which are explicitly defined by the expressions (26) and (28) in Appendix A—see Appendix A also for a detailed derivation of (16) and (17). Notice that in the generic potential $\Phi$ we already include any possible linear coupling with an homogeneous magnetic field. Exploiting the expansion in cumulants of a generic partition functions one easily obtain the formal expression for the structure function and the two-scale correlation function, respectively:

$$\log\langle u_n^p\rangle \propto \sum_k \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k} \langle \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}\rangle_c \, H_n^p(i_1)\, H_n^p(i_2)\cdots H_n^p(i_k); \tag{18}$$

$$\log\langle u_n^p u_{n+m}^q\rangle \propto \sum_k \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k} \langle \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}\rangle_c \, H_{n,n+m}^{p,q}(i_1)\, H_{n,n+m}^{p,q}(i_2)\cdots H_{n,n+m}^{p,q}(i_k), \tag{19}$$

where all connected correlation functions, $\langle \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}\rangle_c$ are calculated at zero external magnetic fields, $\vec{H}=0$. It is easy now to realize that the general expression for, $C_{p,q}^{(m)}$, i.e., the *deviation* from the pure fusion-rules prediction for two-scale correlation function can be expressed as a power series of suitable combination of *external* magnetic fields:

$$\log(C_{p,q}^{(m)}) = \sum_k \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k} \langle \sigma_{i_1}\sigma_{i_2}\cdot\sigma_{i_k}\rangle_c \, M_{i_1, i_2, \ldots, i_k}^{p,q} \tag{20}$$

with

$$M_{i_1, i_2, \ldots, i_k}^{p,q} = \prod_{j=i_1,\ldots,i_k} (H_{n,n+m}^{p,q}(j) + H_n^q(j) - H_{n+m}^q(j) - H_n^{p+q}(j)). \tag{21}$$

Obviously, for a given order, $k$, the different indexes $i_1,\ldots, i_k$ must take values between 1 and $n+m$; indeed, it is sufficient that only one among $i_1,\ldots, i_k$ does not fall between $n$ and $n+m$ to have that the external magnetic fields $H_n^p$, $H_n^q$, $H_{n+m}^q$, $H_{n,n+m}^{p,q}$ vanish and therefore $M_{i_1,\ldots,i_k}^{p,q}=0$.

Expression (20) tell us that the previous log-normal result (14) can be seen as the first contribution, $k=2$, in the above expansion, contribution for $k=1$ being identically zero. It is easy to realize that if the potential is exactly log-normal, only contribution form the $k=2$ term appears, while in the most general case one need to control also three-point correlations, $\langle\sigma\sigma\sigma\rangle_c$ and multi-point correlations.

In Appendix B we develop explicitly the expression of (20) for any order $k$ and also the similar *interesting* expansion obtainable for single scale structure functions (18). It is important to remark, that from the latter one also obtain a formal expansion of $\zeta(p)$ exponents in power of $p$:

$$\zeta(p) = \sum_{j>0} c_j p^j \tag{22}$$

where the set of $c_j$ are connected to the choice of the interacting potential, $\Phi$. In the case of a log-normal potential we nay write the explicit expression of scaling exponents in the limit of small enough scales, i.e., for $n$ large enough:

$$\zeta(p) = p\frac{1}{n}\sum_{i=1}^{n}\langle\sigma_i\rangle_c - \frac{p^2\log(2)}{2n}\sum_{i=1}^{n}(\langle\sigma_1\sigma_i\rangle_c + \langle\sigma_2\sigma_i\rangle_c \cdots \langle\sigma_n\sigma_i\rangle_c). \quad (23)$$

The above expression tells us that the renormalization of exponents due to the appearance of correlation among multipliers is linked to the *magnetic susceptibility*, i.e., to the integral of the connected two-point correlation for all scale separations. Obviously, in the simple case of pure uncorrelated log-normal process with $\langle\sigma_i\sigma_j\rangle_c = \delta_{i,j}\cdot c^2$ and $\langle\sigma_i\rangle_c = h_0$, one finds the well known result:

$$\zeta(p) = ph_0 - \tfrac{1}{2}p^2c^2\log(2). \quad (24)$$

## 5. CONCLUSIONS

A Gibbs-like approach for single time multi-scale correlation functions in a class of random multiplicative process with non-trivial time dependencies has been investigated. We have shown that there exists an *optimal* log-normal Gibbs-like measure able to reproduce with high accuracy the effects induced by the temporal dynamics on the single-time correlation functions. We have explicitly calculated the expression of the most general two-scale correlation functions $\langle u_n^p u_{n+m}^q\rangle$ in the log-normal approximation. We have also shown that the log-normal result can be seen as the first order of a formal cumulant expansion obtained for a completely general potential of interactions between spins (multipliers). Within this formalism, also scaling exponents, $\zeta(p)$ have a simple power-law expansion in terms of the order of the moment, $p$. The first two terms in this expansion coincides with the usual quadratic log-normal expression.

Qualitatively, we expect that very similar results can be obtained in order to describe the multipliers statistics of more realistic models as the case of shell models. The qualitative similar behavior shown in Figs. 2 and 3 indeed is a good evidence that the stochastic process here studied mimics quite well the shell model dynamics. The presence of non-trivial phase-phase correlations and phase-amplitude correlations in shell models is the major difficulties to overcome if one wants to apply in a quantitative way the Gibbs approach in this case. To proceed on this route one needs to introduce some *spin* variables describing phase fluctuations and some new potential describing phase-phase interactions and phase-moduli interactions (see also ref. 7). Also, in shell models higher than second order connected correlation functions appears, as can be easily checked numerically, and therefore the log-normal approximation must be meant only as the first order term in the cumulant expansion as previously discussed.

## APPENDIX A

In order to compute analytically the expression for any correlation function, it is useful to realize that the calculation of either structure functions or multi-scale correlation function can be reduced to the calculation of a particular partition function with a *suitable* non-homogeneous magnetic field, $\mathscr{Z}(\vec{H}, p, q)$, where with $\vec{H}$ we intend the one-dimensional vector whose $N$ components are given by the magnetic field in each site: $\vec{H} = (h_1, h_2,..., h_N)$. In particular, from (11) and (9) one gets for the structure function:

$$\langle u_n^p \rangle \sim \int \prod_{i=1}^{N} d\sigma_i \, \exp^{h_0/c^2 \sum_{i=1}^{N} \sigma_i - p \log(2) \sum_{i=1}^{n} \sigma_i - \frac{1}{2} \sum_{i,j} A_{i,j} \sigma_i \sigma_j}. \tag{25}$$

The extra term in the exponential $p \log(2) \sum_{i=1}^{n} \sigma_i$ can be seen as an additional, position dependent, magnetic field of intensity $p$. Therefore, the structure function $\langle u_n^p \rangle$ is proportional to the original partition function with a modified magnetic field

$$\langle u_n^p \rangle \propto \mathscr{Z}(\vec{H}_n^p) \propto \exp^{\frac{1}{2}(\vec{H}_n^p + \vec{h}_0, \hat{A}^{-1}, \vec{H}_n^p + \vec{h}_0)},$$

where $\vec{H}_n^p$ is a vector with components given by:

$$H_n^p(i) = -\theta(n-i)\, p \log(2), \tag{26}$$

where we have introduced the Heaviside function, $\theta(x)$. In this notation, the non-homogeneous magnetic field acts from the integral scale $i = 1$ up to the inertial scale $i = n$. It is easy to realize, that similar expressions can be derived for the most general multi-scale correlation functions. In particular, the two scale correlation function, $\langle u_n^p u_{n+m}^q \rangle$ can be also calculated via a new partition function with a modified, site-dependent, magnetic field:

$$\langle u_n^p u_{n+m}^q \rangle \propto \mathscr{Z}(\vec{H}_{n,n+m}^{p,q}) \sim \exp^{\frac{1}{2}(\vec{H}_{n,n+m}^{p,q} + \vec{h}_0, \hat{A}^{-1}, \vec{H}_{n,n+m}^{p,q} + \vec{h}_0)}, \tag{27}$$

where with $\vec{H}_{n,n+m}^{p,q}$ we denote the magnetic field vector whose $i$th component is given by:

$$H_{n,n+m}^{p,q}(i) = -(\theta(n-i)\, p + \theta(n+m-i)\, q) \log(2). \tag{28}$$

Now, it is long but simple to show with algebraic manipulation of previous expressions, that the prefactor, $C^{(m)}_{p,q}$ defining the deviation of $\langle u^p_n u^q_{n+m}\rangle$ from the exact fusion rules prediction is given by:

$$C^{(m)}_{p,q} = \frac{\langle u^p_n u^q_{n+m}\rangle\langle u^q_n\rangle}{\langle u^q_{n+m}\rangle\langle u^{p+q}_n\rangle} = \exp^{\mathcal{T}^m_{p,q}} \tag{29}$$

with

$$\mathcal{T}^m_{p,q} = (\log 2)^2\, pq\left(\sum_{i=1}^{n} \langle\sigma_i\sigma_{n+1}\rangle_c + \langle\sigma_i\sigma_{n+2}\rangle_c + \cdots + \langle\sigma_i\sigma_{n+m}\rangle_c\right), \tag{30}$$

as reported in the text.

## APPENDIX B

Let us here analyzed in more details the expression given in the body of the paper for the normalized two-scale correlation function:

$$\log C^{(m)}_{p,q} = \sum_k \frac{1}{k!} \sum_{i_1,i_2,\ldots,i_k} \langle\sigma_{i_1}\sigma_{i_2}\cdot\sigma_{i_k}\rangle_c\, M^{p,q}_{i_1,i_2,\ldots i_k} \tag{31}$$

with

$$M^{p,q}_{i_1,i_2,\ldots,i_k} = \prod_{j=i_1,\ldots,i_k} (H^{p,q}_{n,n+m}(j) + H^q_n(j) - H^q_{n+m}(j) - H^{p+q}_n(j)). \tag{32}$$

It is easy check that for $\mathbf{k=1}$ we may distinguish two cases (i) if $1\leqslant i_1\leqslant n$

$$M^{p,q}_{i_1} = \log^2(-(p+q)-q+q+(p+q)) = 0; \tag{33}$$

(ii) if $n<i_1\leqslant n+m$:

$$M^{p,q}_{i_1} = \log 2(-q-0+q+0) = 0. \tag{34}$$

Therefore we may conclude that the first order contribution in the cumulant expansion is identically zero.

For $\mathbf{k=2}$, the only non-vanishing contributions are those with $1\leqslant i_1\leqslant n$ and $n<i_2\leqslant n+m$, which give:

$$M^{p,q}_{i_1,i_2} = (\log 2)^2\,(q(p+q)-q^2) = (\log 2)^2\, pq \tag{35}$$

and after a permutation between $i_1$ and $i_2$ we have:

$$\log(C_{p,q}^{(m)})|_{k=2} = (\log 2)^2\, pq \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \sigma_i \sigma_{n+j} \rangle_c \tag{36}$$

which coincide with the log-normal contribution.

For orders larger than 2, things becomes more complex, for example for $\mathbf{k}=\mathbf{3}$ we have two cases with non-vanishing contributions:

- $1 \leqslant i_1 \leqslant n$ and $n < i_2, i_3 \leqslant n+m$ leading to:

$$M_{i_1,i_2,i_3}^{p,q} = (\log 2)^3\, (-(p+q)\, q^2 + q^3) = -(\log 2)^3\, (pq^2) \tag{37}$$

- $1 \leqslant i_1, i_2 \leqslant n$ and $n < i_3 \leqslant n+m$ leading to:

$$M_{i_1,i_2,i_3}^{p,q} = (\log 2)^3\, (-q(p+q)^2 + q^3) = (\log 2)^3\, (-qp^2 - 2pq^2). \tag{38}$$

By considering all indexes permutation we get:

$$\begin{aligned}\log(C_{p,q}^{(m)})|_{k=3} = -(\log 2)^3 \bigg( & pq^2 \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{s=1}^{m} \langle \sigma_i \sigma_{n+j} \sigma_{n+s} \rangle_c \\ & + (2pq^2 + qp^2) \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{m} \langle \sigma_i \sigma_j \sigma_{n+s} \rangle_c \bigg).\end{aligned} \tag{39}$$

It is now possible to explicitly write down the generalization to any order $\mathbf{k}$. If we consider the indexes $i_1, \ldots, i_k$ there are $n_1$ indexes between 1 and $n$; $n_2$ indexes between $n$ and $n+m$; with obviously $n_1 + n_2 = k$. We therefore have:

$$\begin{aligned}\log(C_{p,q}^{(m)})|_k = & \sum_{n_1+n_2=k} (-\log 2)^k \, \{[(p+q)^{n_1}\, (+q)^{n_2} - (q)^k] \\ & \cdot \sum_{j_1=1}^{n} \cdots \sum_{j_{n_1}=1}^{n} \sum_{s_1=1}^{m} \cdots \sum_{s_{n_2}=1}^{m} \langle \sigma_{j_1} \cdots \sigma_{j_{n_1}} \sigma_{s_1+n} \cdots \sigma_{s_{n_2}+n} \rangle_c \Big\}\end{aligned} \tag{40}$$

with $n_1 > 0$ and $n_2 > 0$.

Let us now discuss how to obtain the expansion of scaling exponents $\zeta(p)$ in power of $p$. Starting from their definition:

$$\langle u_n^p \rangle \sim k_n^{-\zeta(p)}, \qquad k_n = 2^n \tag{41}$$

we have, using (19):

$$\zeta(p) = -\frac{1}{n \log 2} \sum_k \frac{1}{k!} \sum_{i_1,\ldots,i_k} \langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle_c \, G_{i_1,\ldots,i_k}^{p} \tag{42}$$

where:

$$G^{p}_{i_1,\ldots,i_k} = \prod_{j=i_1,\ldots,i_k} H^{p}_{n}(j) = (-p\log 2)^k, \tag{43}$$

because all indexes $i_1,\ldots,i_k$ must be between 1 and $n$ otherwise $G^{p}_{i_1,\ldots,i_k} = 0$. Therefore, we obtain the expression cited in the body of the paper:

$$\zeta(p) = \sum_{j>0} c_j p^j \tag{44}$$

with $c_j$ defined in terms of the microscopic potential $\Phi$.

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